

DIAMETER OF RAMANUJAN GRAPHS AND RANDOM CAYLEY GRAPHS WITH NUMERICS

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ABSTRACT. For an infinite family of $(p+1)$ -regular LPS Ramanujan graphs, we show that the diameter of these graphs is greater than or equal to $\lfloor \frac{4}{3} \log_p(n) \rfloor$, where p is an odd prime number and n is the number of vertices. On the other hand, for any k -regular Ramanujan graph we show that the distance of only tiny fraction of all pairs of vertices is greater than $(1 + \epsilon) \log_{k-1}(n)$. We also have some numerical experiments for LPS Ramanujan graphs and random Cayley graphs which suggest that the diameters are asymptotically $\frac{4}{3} \log_{k-1}(n)$ and $\log_{k-1}(n)$, respectively. These are consistent with Sarnak's expectation on the covering exponent of universal quantum gates and our conjecture for the optimal strong approximation for quadratic forms in 4 variables.

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1. INTRODUCTION

1.1. Motivation. The diameter of any k -regular graph with n vertices is bounded from below by $\log_{k-1}(n)$ trivially. While the diameter of a general connected k -regular graph may be as large as a scalar multiple of the number of vertices n , it is known that the diameter of any k -regular Ramanujan graph is bounded from above by $2(1 + \epsilon) \log_{k-1}(n)$ [Sar90]. Lubotzky, Phillips and Sarnak constructed a family of $(p+1)$ -regular Ramanujan graphs $X^{p,q}$ [Sar90], where p and q are prime numbers. $X^{p,q}$ is the Cayley graph of $PGL(2, \frac{\mathbb{Z}}{q\mathbb{Z}})$ with $p+1$ explicit generators. Their construction can be modified for every integer q ; see [DSV03] or [Lub10]. It was expected that the diameter of the LPS Ramanujan graphs to be bounded from above by $(1 + \epsilon) \log_{k-1}(n)$; see [Sar90, Chapter 3]. However, we show that the

diameter of a family of $p + 1$ -regular LPS Ramanujan graphs are greater than or equal to

$$(1.1) \quad \lfloor \frac{4}{3} \log_p(n) \rfloor.$$

While there are points x and y whose distances is large in a LPS Ramanujan graph, we prove that the distance of almost all pairs of vertices in any k -regular Ramanujan graph G is less than $(1+\epsilon) \log_{k-1}(n)$. In fact, we prove a stronger result, we show that for every vertex x in a k -regular Ramanujan graph G the number of points which we can't visit by exactly l steps, where $l > (1 + \epsilon) \log_{k-1}(n)$, is less than $n^{1-\epsilon}$. So the number is exponentially decaying. In particular, it also recovers $2(1 + \epsilon) \log_{k-1}(n)$ as an upper bound on the diameter of k -regular Ramanujan graph. Furthermore, we give some numerical datas for two families of 6-regular graphs. The first family of graphs are the 6-regular LPS Ramanujan graphs and we denote them by $X^{5,q}$. The second family are the 6-regular random Cayley graphs $PSL(2, \frac{\mathbb{Z}}{q\mathbb{Z}})$, i.e. the Cayley graphs that are constructed by 3 random generators of $PSL(2, \frac{\mathbb{Z}}{q\mathbb{Z}})$ and their inverses $\{s_1^\pm, s_2^\pm, s_3^\pm\}$. We denote these graphs by Z^q . The numerical experiments suggest that the diameter of the (number theoretic) LPS Ramanujan Graphs is asymptotic to

$$(1.2) \quad \frac{4}{3} \log_5(n).$$

This is consistent with our conjecture on the optimal strong approximation for quadratic forms in 4 variables [Tal15]. On the other hand, the numerical data suggests that the diameter of the random Cayley graph equals that of a random 6-regular graph [BFdlV82], that is

$$(1.3) \quad \log_5(n).$$

These are consistent with Sarnak's expectation. Sarnak suggested that the covering exponent of a thin set of gates may achieve the trivial lower bound 1 and the covering exponent of arithmetic gates are $4/3$ [Sar15]. The archimedean analog of our question has been discussed there; see [Sar15]. For instance, the diameter of the LPS Ramanujan Graphs is bounded from above by $(1 + \epsilon) \frac{4}{3} \log_{k-1}(n)$ is analogous to the covering exponent being bounded from above by $\frac{4}{3}$. This question has been raised by Sarnak in his notes and is related to the theory of quadratic Diophantine equations; see [Tal15]. Sarnak showed that the almost all covering exponent is 1; see [Sar15, Page 28]. Our Theorem 1.3 is the p -adic analogue, i.e. we show that almost all pair of points have a distance less than $(1 + \epsilon) \log_{k-1}(n)$. In a recent paper [LP15], Peres and Lubetzky show the simple random walk exhibits cutoff on Ramanujan Graphs. As a result they give a more detailed version of Theorem 1.3. In a similar work, for the family of LPS bipartite Ramanujan graphs, Biggs and Boshier determined the asymptotic behavior of the girth of these graphs; see [BB90]. They showed that the girth is asymptotic to

$$\frac{4}{3} \log_{k-1}(n).$$

1.2. Statement of results. We begin by the description of the LPS Ramanujan graphs. The idea of the construction is coming from number theory, i.e. generalized Ramanujan conjecture. More precisely, we consider the symmetric space $PGL(2, \mathbb{Q}_p)/PGL(2, \mathbb{Z}_p)$ which can be identified with a regular $(p+1)$ -infinite tree.

We note that $PGL(2, \mathbb{Z}[1/p])$ acts from the right on $PGL(2, \mathbb{Q}_p)/PGL(2, \mathbb{Z}_p)$. The generalized Ramanujan conjecture, which is a theorem in this case, implies that the quotient of $PGL(2, \mathbb{Q}_p)/PGL(2, \mathbb{Z}_p)$ by any congruence subgroup of $PGL(2, \mathbb{Z}[1/p])$ which is a $p+1$ -regular graph is a Ramanujan graph. By considering an appropriate congruence subgroup of $PGL(2, \mathbb{Z}[1/p])$ we can identify the quotient of this symmetric space with a Cayley graph of $PSL(2, \frac{\mathbb{Z}}{m\mathbb{Z}})$ with $p+1$ generators. These are LPS Ramanujan graphs. We show that the diameter of LPS Ramanujan graphs are greater than

$$\frac{4}{3} \log_p(n) - 4 \log_p\left(\frac{m}{q}\right) - \frac{2}{3} \log_p 2,$$

provided that $q|m$, where q is a prime power and $q \neq m$. Note that these graphs are congruence cover of LPS Ramanujan graphs $X^{p,q}$.

In what follows, we give an explicit description of LPS Ramanujan graphs in terms of Cayley graphs of $PSL(2, \frac{\mathbb{Z}}{m\mathbb{Z}})$. Assume that q is a prime number and $q|m$ where m is an integer and -1 is quadratic residue mod m . Let p be a prime number such that $p \equiv 1 \pmod{4}$ and p is quadratic residue mod m . We denote the representatives of square roots of -1 and p mod m by i and \sqrt{p} , respectively. We are looking at the integral solutions $\alpha = (x_0, x_1, x_2, x_3)$ of the following diophantine equation

$$(1.4) \quad x_0^2 + x_1^2 + x_2^2 + x_3^2 = p,$$

where $x_0 > 0$ and is odd and x_1, x_2, x_3 are even numbers. There are exactly $p+1$ integral solutions with such properties. To each such integral solution α , we associates the following matrix α in $PSL(2, \frac{\mathbb{Z}}{m\mathbb{Z}})$:

$$(1.5) \quad \alpha := \frac{1}{\sqrt{p}} \begin{bmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 I' x_1 \end{bmatrix}.$$

This gives us $p+1$ matrices in $PSL(2, \frac{\mathbb{Z}}{m\mathbb{Z}})$. Lubotzky [Lub10, Theorem 7.4.3] showed that they generate $PSL(2, \frac{\mathbb{Z}}{m\mathbb{Z}})$ and the associated Cayley graph is a non-bipartite Ramanujan graph. We denote this graph by $X^{p,m}$. Furthermore, Lubotzky showed that

- $\text{diam } X^{p,m} \leq 2 \log_p(n) + 2 \log_p 2 + 1.$
- $\text{girth } X^{p,m} \geq \frac{2}{3} \log_p(n) - 2 \log_p(2).$

Our first theorem shows that either the distance between the identity matrix I and $I' := \begin{bmatrix} 1 & q \\ q & 1 \end{bmatrix}$ or between I and $W := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ in $PSL(2, \frac{\mathbb{Z}}{m\mathbb{Z}})$ is larger than

$$\left(\frac{4}{3}\right) \log_p(n) - 4 \log_p\left(\frac{m}{q}\right) - \frac{2}{3} \log_p 2.$$

As a result,

$$(1.6) \quad \left(\frac{4}{3}\right) \log_p(n) - 4 \log_p\left(\frac{m}{q}\right) - 1/3 \log_p 2 \leq \text{diam}(X^{p,m}).$$

We state our first theorem which we prove in Section 2.

Theorem 1.1. *Let m be an integer such that -1 is quadratic residue mod m . Assume that q is an odd prime power such that $q|m$ and $q \neq m$. Let p be an odd prime number, such that p is quadratic residue mod m . Let $X^{p,m}$ be the LPS*

Ramanujan graph. Then we have the following lower bound on the diameter of the LPS graph $X^{p,m}$

$$(1.7) \quad \left(\frac{4}{3}\right) \log_p(n) - 4 \log_p\left(\frac{m}{q}\right) - \frac{2}{3} \log_p 2 \leq \text{diam}(X^{p,m}).$$

Corollary 1.2. *Let p and q be prime numbers that are congruent to 1 mod 4 and $p > 1250$. Then the diameter of the LPS Ramanujan graph $X^{p,5q^k}$ for any integer k is greater than or equal to*

$$(1.8) \quad \lfloor \frac{4}{3} \log_p n \rfloor$$

By contrast, we prove that the distance of almost all pairs of vertices in any k -regular Ramanujan graph G is less than $(1 + \epsilon) \log_{k-1}(n)$. We use the Ramanujan bound on the nontrivial eigenvalues of the adjacency matrix to prove the distance of almost all pairs of vertices is less than $(1 + \epsilon) \log_k(n)$. The archimedean version of this problem has been discussed in Sarnak's letter to Scott Aaronson and Andy Pollington [Sar15, Page 28]. More precisely, we prove the following stronger result in Section 3:

Theorem 1.3. *Let G be a k -regular Ramanujan graph and fix a vertex $x \in V(G)$. Let R be an integer such that $R > (1 + \epsilon) \log_{k-1}(n)$. Define $M(x, R)$ to be the set of all vertices $y \in G$ such that there is no path from x to y with length R (we allow to pass from an edge multiple times but not immediately after). Then,*

$$(1.9) \quad |M(x, l)| \leq n^{1-\epsilon} (1 + R)^2.$$

1.3. Outline of the paper. In Section 2, we prove Theorem 1.1. The proof uses some elementary facts in Number Theory and is short. In Section 3, we prove Theorem 1.3. As a corollary, we prove that the distance of almost all pair of vertices is less than

$$(1 + \epsilon) \log_{k-1}(n).$$

We use the Chebyshev's inequality by giving an upper bound for the variance. We use the Ramanujan bound on the eigenvalues of the adjacency matrix of the graph to give an upper bound on the variance. Finally in Section 4, we compute the diameter of two families of 6-regular graphs. From our numerical experiments, we expect that the diameter of the LPS Ramanujan graphs [LPS88] is asymptotic to

$$(1.10) \quad \frac{4}{3} \log_p(n).$$

We define a random 6 regular Cayley graph Z^q , by considering the Cayley graph of $PSL(2, \frac{\mathbb{Z}}{q\mathbb{Z}})$ relative to the generating set $S = \{s_1^\pm, s_2^\pm, s_3^\pm\}$, where s_1, s_2, s_3 are random elements of $PSL(2, \frac{\mathbb{Z}}{q\mathbb{Z}})$. From the numerical experiments, we show that in fact the random Cayley graph has a shorter diameter and break the $\frac{4}{3} \log_5 n$ lower bound for the LPS Ramanujan graphs. For example, We obtained a sample from the random Cayley graph of $PSL(2, F_{181})$, such that

$$(1.11) \quad \text{diam}(Z_{181}) < 1.18 \log_5 n.$$

We expect that the diameter of the random Cayley graph would be as small as possible. Precisely, for $\epsilon > 0$

$$(1.12) \quad \text{diam}(Z^q) \leq (1 + \epsilon) \log_5(n) + c_\epsilon, \text{ almost surely as } q \rightarrow \infty.$$

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2. LOWER BOUND FOR THE DIAMETER OF THE RAMANUJAN GRAPHS

In the rest of this section, we give a proof of Theorem 1.1.

Proof. Recall that $W := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $I' := \begin{bmatrix} 1 & q \\ q & 1 \end{bmatrix}$. We show that

$$(2.1) \quad \max(\text{dist}(I, I'), \text{dist}(I, W)) \geq \frac{4}{3} \log_p(n) - 4 \log_p\left(\frac{m}{q}\right) - \frac{2}{3} \log_p 2.$$

Assume the contrary that

$$(2.2) \quad \max(\text{dist}(I, I'), \text{dist}(I, W)) < \frac{4}{3} \log_p(n) - 4 \log_p\left(\frac{m}{q}\right) - \frac{2}{3} \log_p 2.$$

Since $n = \frac{(m-1)m(m+1)}{2}$, the above assumption is equivalent to

$$(2.3) \quad \max(\text{dist}(I, I'), \text{dist}(I, W)) < \log_p \frac{q^4}{4}$$

$\text{dist}(I, I') < \log_p \frac{q^4}{4}$ gives us a solution to the following diophantine equation

$$(2.4) \quad a^2 + b^2 + c^2 + d^2 = p^k,$$

where $b \equiv c \equiv d \equiv 0 \pmod{2q}$ and $a \equiv 1 \pmod{2}$. At least one of b, c, d is nonzero. From this we deduce that

$$(2.5) \quad a^2 \equiv p^k \pmod{q^2} \text{ and } 4q^2 \leq p^k.$$

We consider two cases: k even and k odd.

If k is even and $k = 2t$. From 2.5 we deduce that

$$(2.6) \quad a \equiv \pm p^t \pmod{q^2}.$$

If $p^t \geq \frac{q^2}{2}$,

$$(2.7) \quad \text{dist}(I, I') = 2t \geq \log_p \frac{q^4}{4},$$

a contradiction. Consequently, $p^t < \frac{q^2}{2}$. Since $a \not\equiv \pm p^t$, we deduce that

$$(2.8) \quad a = p^t + lq^2 \text{ for } l \neq 0.$$

Therefore

$$(2.9) \quad \|a\| \geq \frac{1}{2} q^2.$$

Hence,

$$(2.10) \quad p^{2t} \geq \frac{q^4}{4}, \text{ and so } \text{dist}(I, I') = 2t \geq \log_p \frac{q^4}{4}.$$

a contradiction. Hence k is odd and $k = 2t + 1$.

We want to use a similar argument to show that $\text{dist}(I, W) = 2t_0 + 1$ is an odd number. $\text{dist}(I, W) < \frac{4}{3} \log_p(n)$ gives us a solution to the following diophantine equation

$$(2.11) \quad a^2 + b^2 + c^2 + d^2 = p^k.$$

Where $b \equiv a \equiv d \equiv 0 \pmod{q}$ and $c \equiv 0 \pmod{2}$. Since a is odd, then $q \leq \|a\|$. We deduce that

$$(2.12) \quad c^2 \equiv p^k \pmod{q^2} \text{ and } q^2 \leq p^k.$$

We consider two cases: k even and k odd.

If k is even and $k = 2t$, from 2.12 we deduce that

$$(2.13) \quad c \equiv \pm p^t \pmod{q^2}.$$

If $p^t \geq \frac{q^2}{2}$,

$$(2.14) \quad \text{dist}(I, W) = 2t \geq \log_p \frac{q^4}{4},$$

a contradiction. Consequently, $p^t < \frac{q^2}{2}$. Since c is even, then $c \neq \pm p^t$. We deduce that

$$(2.15) \quad c = \pm p^t + lq^2 \text{ for } l \neq 0.$$

Therefore,

$$(2.16) \quad c \geq \frac{1}{2}q^2.$$

Hence,

$$(2.17) \quad \begin{aligned} p^{2t} &\geq \frac{1}{4}q^4, \\ \text{dist}(I, W) = 2t &\geq \log_p \frac{q^4}{4}. \end{aligned}$$

This is a contradiction. Therefore $k = 2t_0 + 1$ for some t_0 .

We now investigate the case where

$$\text{dist}(I, I') = 2t + 1 < \log_p \frac{q^4}{4}$$

and

$$\text{dist}(I, W) = 2t_0 + 1 < \log_p \frac{q^4}{4}.$$

$\text{dist}(I, I') = 2t + 1$ gives us a solution to the following diophantine equation

$$(2.18) \quad a^2 + b^2 + c^2 + d^2 = p^{2t+1} < \frac{q^4}{4}.$$

Where $b \equiv c \equiv d \equiv 0 \pmod{2q}$ and $a \equiv 1 \pmod{2}$. At least one of b, c, d is nonzero. Hence

$$(2.19) \quad 4q^2 < p^{2t+1} < q^4.$$

$\text{dist}(I, W) = 2t_0 + 1 < \log_p \frac{q^4}{4}$, gives us a solution to the following diophantine equation

$$(2.20) \quad a_0^2 + b_0^2 + c_0^2 + d_0^2 = p^{2t_0+1} < q^4/4.$$

Where $b_0 \equiv a_0 \equiv d_0 \equiv 0 \pmod{q}$ and $a_0 \equiv 1 \pmod{2}$. From 2.18 and 2.20 we deduce that

$$(2.21) \quad \begin{aligned} a^2 &\equiv p^{2t+1} \pmod{q^2} \text{ and } a \text{ is odd } a < p^{t+1/2} < q^2/2, \\ c^2 &\equiv p^{2t_0+1} \pmod{q^2} \text{ and } c \text{ is even } c < p^{t_0+1/2} < q^2/2. \end{aligned}$$

WLOG assume that $t_0 > t$, from 2.21 we deduce that

$$(2.22) \quad \pm ap^{t_0-t} = c.$$

However, this is incompatible with the parities of a and c . Hence, we conclude Theorem 1.1. \square

3. VISITING ALMOST ALL POINTS AFTER $(1 + \epsilon) \log_{k-1}(n)$ STEPS

In this section, we show that if we pick two random points from a k -regular Ramanujan graph G , then almost surely they have a distance less than

$$(3.1) \quad (1 + \epsilon) \log_{k-1}(n).$$

The idea is to use the spectral gap of the adjacency matrix of the Ramanujan graphs to prove an upper bound on the variance. A similar strategy has been implemented by Sarnak ; see [Sar15, Page 28].

Let $A(x, y)$ be the adjacency matrix of the Ramanujan graph G , i.e.

$$(3.2) \quad A(x, y) := \begin{cases} 1 & \text{if } x \equiv y \\ 0 & \text{otherwise} \end{cases}.$$

Since $A(x, y)$ is a symmetric matrix, so it is diagonalizable. We can write the spectral expansion of this matrix by the set of its eigenfunctions. Namely,

$$(3.3) \quad A(x, y) = \frac{k}{\|G\|} + \sum_j \lambda_j \phi_j(x) \phi_j(y),$$

where $\{\phi_j\}$ is the orthonormal basis of the nontrivial eigenfunctions with eigenvalues $\{\lambda_j\}$ for the adjacency matrix $A(x, y)$. Since we assumed that G is a Ramanujan graph, then $|\lambda_j| \leq 2\sqrt{k-1}$. We change the variables and write

$$(3.4) \quad \lambda_j = 2\sqrt{k-1} \cos \theta_j.$$

We define $S(R) := (k-1)^{\frac{R}{2}} U_R(\frac{A}{2\sqrt{k-1}})$, where $U_R(x)$ is the Chebyshev polynomial of the second kind, i.e.

$$(3.5) \quad U_R(x) := \frac{\sin((R+1) \arccos x)}{\sin(\arccos x)}.$$

The following is the spectral expansion of $S(R)$:

$$(3.6) \quad S(R)(x, y) := \frac{(k-1)^{\frac{R}{2}} U_R(\frac{k}{2\sqrt{k-1}})}{\|G\|} + \sum_j (k-1)^{\frac{R}{2}} U_R(\frac{\lambda_j}{2\sqrt{k-1}}) \phi_j(x) \phi_j(y),$$

Remark 3.1. Note that if we lift the linear operator $S(R)$ to the universal covering space of the k -regular graph G , (which is an infinite k -regular tree), then $S(R)$ is

the linear operator, which takes the average of a function on a sphere with radius R . Namely,

$$(3.7) \quad S(R)f(x) := \sum_{y, \text{dist}(x,y)=R} f(y).$$

From the formula for the kernel of $S(R)$ given in 3.6, we obtain

$$(3.8) \quad S(R)(x, y) = \frac{k(k-1)^{R-1}}{\|G\|} + \sum_j (k-1)^{\frac{R}{2}} \frac{\sin((R+1)\theta_j)}{\sin \theta_j} \phi_j(x) \phi_j(y).$$

We calculate the variance over y . For $i \neq j$, we have $\sum_{y \in G} \phi_i(y) \phi_j(y) = 0$ and $\sum_{y \in G} \phi_i(y)^2 = 1$, So only the diagonal terms remain in the following summation:

$$(3.9) \quad \begin{aligned} \text{Var}(x) &:= \sum_{y \in G} \|S(R)(x, y) - \frac{k(k-1)^{R-1}}{\|G\|}\|^2 \\ &= \sum_j (k-1)^R \frac{(\sin(R+1)\theta_j)^2}{(\sin \theta_j)^2} \phi_j(x)^2. \end{aligned}$$

Since $\{\phi_j\}$ is an orthonormal basis, we have

$$(3.10) \quad 1 = \sum_{y \in G} \delta(x, y) dy = \frac{1}{\|G\|} + \sum_j \phi_j(x)^2.$$

We also have the following trivial trigonometric inequality, which is derived from the geometric series summation formula :

$$(3.11) \quad \left| \frac{\sin(R+1)\theta}{\sin \theta} \right| = \left| \sum_{j=0}^R e^{i\theta} \right| \leq R+1.$$

From 3.10 and 3.11, we obtain

$$(3.12) \quad \text{Var} \leq (R+1)^2 (k-1)^R.$$

We define

$$(3.13) \quad M := \{y : S(R)(x, y) = 0\}.$$

Note that M is the set of all vertices $y \in G$, such that there is no path from x to y with length R . Therefore, this is exactly the set $M(x, R)$ as defined in the Theorem 1.3. By the definition of the Var given in 3.9,

$$(3.14) \quad \|M\| \left| \frac{k(k-1)^{R-1}}{\|G\|} \right|^2 \leq \text{Var}.$$

From 3.14 and 3.12, we have

$$(3.15) \quad \|M\| \|(k-1)^R\| < \|G\|^2 (R+1)^2.$$

If we choose $R > (1+\epsilon) \log_{k-1}(n)$, Hence

$$(3.16) \quad \|M\| \leq n^{1-\epsilon} (1+R)^2.$$

Therefore, we conclude the Theorem 1.3.

4. NUMERICAL RESULTS

In this section, we present our numerical experiments for the diameter of the 6-regular LPS Ramanujan graphs $X^{5,q}$ and 6-regular random Cayley graphs Z^q .

The construction of LPS Ramanujan graphs $X^{5,q}$ requires that 5 and -1 to be quadratic residues mod q . From the reciprocity law we deduce that all the prime factors of q are congruent to 1 or 9 mod 20. We give the diameter of the LPS Ramanujan graphs $X^{5,q}$ for $1 \leq q \leq 229$ in the following table:

q	number of Vertices of $X^{5,q}$	Diameter	$\frac{\text{diam}}{\log_5 n}$
29	12180	8	1.36
41	34440	9	1.38
61	113460	9	1.24
89	352440	11	1.38
101	515100	11	1.34
109	647460	11	1.32
149	1653900	12	1.34
181	3375540	14	1.51
229	6004380	13	1.34

TABLE 1. LPS Ramanujan graphs $X^{5,q}$

We note that $\frac{\text{diam}}{\log_5 n}$ are clustered around $\frac{4}{3}$ for these LPS Ramanujan graphs. We expect that $\frac{\text{diam}}{\log_5 n}$ converges to $\frac{4}{3}$ as $q \rightarrow \infty$.

Finally, we give our numerical experiments for the diameter of the 6-regular random Cayley graphs $PSL(2, \frac{\mathbb{Z}}{q\mathbb{Z}})$. To compare the diameter of the random Cayley graphs with that of the LPS Ramanujan graphs given above, we choose the same set of integers q . We generate 8 random samples for each q , and we give the averaged ratio $\frac{\text{diam}}{\log_5 n}$ in the last column of the following table:

q	number of Vertices of Z^q	Diameter	$\frac{\text{diam}}{\log_5 n}$
29	12180	$8 \times 6 9 \times 2$	1.50
41	34440	$9 \times 4 8 \times 4$	1.30
61	113460	$9 \times 5 10 \times 3$	1.29
89	352440	$10 \times 5 11 \times 3$	1.30
101	515100	$10 \times 5 11 \times 3$	1.26
109	647460	$10 \times 4 11 \times 4$	1.26
149	1653900	$11 \times 6 12 \times 2$	1.25
181	3375540	$11 \times 3 12 \times 5$	1.24
229	6004380	12×8	1.23

TABLE 2. Random Cayley graphs $PSL_2(\frac{\mathbb{Z}}{q\mathbb{Z}})$ with 6 generators

$8 \times 6 9 \times 2$ means that 6 of our random samples are 8 and 2 of them are 9. We note that the empirical mean of the ratio $\frac{\text{diam}}{\log_5 n}$ is decreasing in q . By pigeonhole principal, one can show easily that

$$\frac{\text{diam}}{\log_5 n} \geq 1.$$

For random Cayley graphs, we expect that $\frac{\text{diam}}{\log_5 n}$ converges to 1 in probability as $q \rightarrow \infty$.

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